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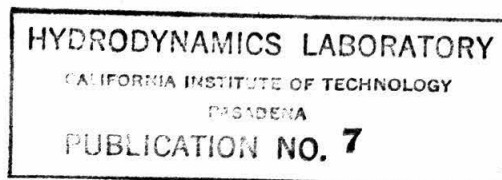
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NOMOGRAM FOR THE SETTLING VELOCITY OF SPHERES

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Of primary importance to many professions is a knowledge of the rate at which a body will fall through a fluid medium. This problem plays an obvious role in the laboratory analysis of sediment, but it is also encountered in such varied fields as meteorology, mining engineering, sanitary engineering, and river hydraulics. The basic problem differs little in these several fields; indeed, many of the latest advances in the subject have resulted from research in still another science - fluid mechanics.

The phenomenon is essentially that of the resistance to motion encountered by an immersed body. The resisting force,  $F$ , was first considered by Newton<sup>1</sup> to depend upon the cross-sectional area of the body,  $A$ , the mass density,  $\rho$ , and the square of the velocity,  $V$ :

$$F = CA\rho\frac{V^2}{2} \quad (1)$$

The factor  $C$  was a numerical coefficient which Newton believed to vary only with the geometrical form of the body and its orientation with respect to the direction of motion.

Newton's basic equation was dimensionally correct, but his reasoning as to the magnitude of  $C$  was physically unsound. This factor depends not only upon shape and orientation, but varies considerably with the viscous characteristics of the motion. The extent to which viscosity may influence such motion, however, depends not only upon the absolute viscosity,  $\mu$ , of the fluid, but also upon the fluid density, the size of the body, and the mean velocity of the relative motion. These four parameters may be combined as a dimensionless ratio called the Reynolds number:

$$R = \frac{VD}{\mu} \rho \quad (2)$$

$R$  is a pure number, and as such will not depend in magnitude upon the dimensional system used - so long as all terms are expressed consistently in units of the same system.  $D$  is a characteristic length of the body; its selection is arbitrary, if homologous lengths are always chosen. The ratio  $\frac{\mu}{\rho}$  is generally denoted by the symbol  $\nu$  and is called the kinematic viscosity.

This Reynolds number actually represents the relative magnitude of the forces involved in mass acceleration and those resulting from viscous shear, as the fluid passes around the body. Thus, the smaller the Reynolds number, the more preponderant the influence of viscosity; conversely, the larger the Reynolds number, the more important are inertial effects and the less the viscous stresses have to do with the total resistance of the motion.

The extent to which variation in the geometrical form of a body will influence the resistance depends in large measure upon the magnitude of  $R$ . Thus, at low Reynolds numbers shape is of secondary importance; but as  $R$  becomes large, the body tends to leave in its course a turbulent wake, the size and intensity of which are governed largely by the form of the longitudinal profile. A stream-lined body, for instance, leaves a wake of minimum size, whereas a very angular profile will produce eddies of greatest magnitude. Since the formation of these eddies represents a continuous drain on the energy of the basic motion, it is evident that the resistance encountered by the body must increase with increasing turbulence in the wake. Experimental measurements indicate that at high Reynolds numbers such resistance may be many times greater than that caused by viscous shear alone.

In order to distinguish between the influence of  $R$  and the influence of shape upon the total resistance, it is expedient to investigate first the effect of variation in  $R$  alone for a given form of body. The sphere is undoubtedly the simplest form that could be chosen, owing to its perfect symmetry. Writing the resistance equation in the general form

$$F = \psi(R) A \frac{V^2}{2} \quad (3)$$

in which the diameter of the sphere is generally adopted as the length parameter in expressing both  $R$  and  $A$ . The factor 2 has been introduced in the denominator, since  $\frac{V^2}{2}$  is equivalent to kinetic energy per unit volume of fluid.  $\psi(R)$  is a mathematical notation indicating that a numerical coefficient  $\psi$  is some function of  $R$ .

Stokes<sup>2</sup> was able to develop an expression for the resistance of spheres at very small values of  $R$ , on the assumption that in this range the mass acceleration of the fluid caused by the motion of the sphere is negligible in comparison with the influence of viscosity. Stokes' relationship may be written as follows, to correspond with the form of Equation (3):

$$F = \frac{24A}{R} \frac{V^2}{2} \quad (4)$$

in which

$$\psi = \frac{24}{R} = \frac{24\eta}{VD} \quad (5)$$

Ladenburg<sup>3</sup> determined the effect of neighboring boundaries upon the resistance to motion; with modification of the second numerical term as noted by Faxen<sup>4</sup>, Ladenburg's correction factor for the presence of vertical walls has the form,

$$(1 + 2.1 \frac{D}{L}) \quad (6)$$

for a sphere of diameter  $D$  falling along the axis of a cylindrical container of diameter  $L$ . Ladenburg's correction is valid only within the range of applicability of Stokes' analysis - up to a Reynolds number of 0.1.

Stokes' equation was extended analytically to higher values of  $R$  by Oseen<sup>5</sup> who obtained the approximate relationship,

$$\psi = \frac{24}{R} (1 + \frac{3}{16}R) \quad (7)$$

Goldstein<sup>6</sup>, in turn, made an exact solution of Oseen's equation with the result

$$\psi = \frac{24}{R} (1 + \frac{3R}{16} - \frac{19}{1280}R^2 + \frac{71}{20480}R^3 - \dots) \quad (8)$$

It is evident that both the Oseen and the Goldstein equations approach that of Stokes as  $R$  becomes very small. Oseen's expression yields good results somewhat beyond that of Stokes, while Goldstein's is trustworthy to approximately  $R = 2$ .

Beyond this limit, however, analytical methods have not yet produced a dependable relationship for the resistance coefficient, and further study of  $\psi$  as a function of  $R$  has necessarily remained empirical. It is evident from Plate I, nevertheless, that a single-valued functional relationship must exist for Reynolds numbers at least as high as 1,000,000, since all experimental data (after Schiller<sup>7</sup>) fall very close to a single curve. The initial deviation from Stokes' law is, of course, due to the ever increasing role played by mass-acceleration during flow around the body. As the growth of a turbulent wake begins to have a marked effect upon the resistance, the deviation becomes more pronounced, and  $\psi$  eventually becomes practically independent of  $R$ . The abrupt drop in the curve at  $R = 250,000$  is due to the onset of turbulence in the boundary layer at the front of the sphere; since the wake is thereupon reduced in size<sup>8</sup>, the value of  $\psi$  is considerably decreased.

Of great importance is the fact that this experimental curve was determined not only by timing the fall (or rise) of spheres through various media but also by measuring the force exerted upon stationary spheres in aeronautical wind-tunnels. The evident agreement of results

obtained by both methods, regardless of considerable variation in fluid properties, is undeniable proof of the validity of the plot.

Treatment of variation in  $\lambda$  with shape, however, is by no means so simple, for a complete definition of shape requires the use of more than one parameter. The additional problem of orientation with respect to direction of motion involves further difficulties, for the inclination of a falling body cannot be controlled in the same manner as that of a body held stationary in a wind-tunnel. For instance, a flat circular plate held normal to the flow attains a minimum  $\lambda$  of 1.12 at  $R = 3600$ , which then remains practically constant as  $R$  increases. On the other hand, a disk falling through a fluid tends to change direction as it descends, the trend of data obtained by Schmiedel indicating a lower coefficient than that for spheres at low Reynolds numbers, but limiting values even higher than those obtained for circular plates in the wind-tunnel. An intermediate form, the cube, yields values of  $\lambda$  fully as high as those for the disk. It is evident that irregularity of form invariably causes a higher resistance to motion - from which it is to be concluded that surface roughness (profile irregularity in a mild form) will show a tendency in the same direction. Needless to say, the effect upon  $\lambda$  will vary with the degree of departure from true sphericity, and with the magnitude of the Reynolds number. In this respect, a fact indicated by Plate I is very pertinent: regardless of shape, all experimental points follow the same general trend as high as  $R = 50$ .

If one attempts to eliminate the shape and orientation factors for various geometrical forms by treatment in terms of equivalent spheres, it must be remembered that a solid of irregular shape may have a volume and hence a weight equal to that of a sphere of one diameter, but a settling velocity of a sphere of quite different dimensions. As indicated by Wadell<sup>9</sup>, the terms "nominal diameter" and "sedimentation diameter" would seem to be appropriate designations for these two parameters. Their determination, however, is a matter beyond the scope of this paper, although whatever the method of dealing with these geometrical factors, one point is clear: the resistance curve for spheres must serve as a basis for comparison, for a more suitable reference cannot be found.

For very low Reynolds numbers, computation of diameter or velocity of fall through use of the familiar Stokes equation involves no further difficulty. But beyond the limit  $R = 0.1$ , use of this equation will begin to introduce error. Moreover, the plot of  $\lambda$  versus  $R$  does not solve this difficulty completely, for unless either  $\lambda$  or  $R$  is known, the determination of diameter or velocity may be accomplished only by means of trial. The preparation of tables to facilitate this computation would be a thankless task, whereas an alignment chart, or nomogram, could eliminate this difficulty entirely.

At least one such nomogram has been published - that by Spronck<sup>10</sup>, which is built around the experimental curve shown in Plate I, after a method indicated by Wadell<sup>11</sup>. Unfortunately, use of this chart is not as simple as one might desire. For research in the Cooperative Laboratory of the Soil Conservation Laboratory, therefore, the writer and Dr. Merit P. White attempted the development of a nomogram which could be read at a single setting of a straight-edge. Such a nomogram may contain only three scales, however, while from the following development it will be evident that the problem involves four pertinent variables. In Equation (3) the force resisting motion,  $F$ , may be replaced by the volume of the sphere multiplied by the difference between the weights per unit volume of sphere and fluid; thus,

$$\frac{\pi D^3}{6} (\gamma_s - \gamma_f) = \frac{\pi D^2}{4} \gamma_f \frac{V^2}{2}$$

Since  $\frac{\gamma_f}{\gamma_f} = g$ , this will become, through division by  $\gamma_f$

$$\left( \frac{\gamma_s}{\gamma_f} - 1 \right) = \frac{3 V^2}{4 g D}$$

The dimensionless quantity on the left is numerically equivalent to the ratio of the specific gravities of solid and fluid, minus 1, and may be designated by the symbol  $S$ .

Within the Stokes range, from Equation (5)

$$S = \frac{18 \eta V}{g D^2}$$

The acceleration of gravity,  $g$ , in c.g.s. units, may be taken as 981 cm/sec<sup>2</sup>; therefore,

$$\frac{V \eta}{S D^2} = 54.5 \quad (9)$$

Since an alignment chart of the type desired is limited to three parallel scales, it is evident that one of the four variables must appear in product form with one or more of the remaining three. Inasmuch as  $S$  varies over a limited range and is a factor that is practically always known, it would seem most expedient to select it as the multiplier. Equation (9) will then have the form

$$\frac{V(S \eta)}{(S D)^2} = 54.5 \quad (10)$$

for which the three scales will indicate the quantities  $V$ ,  $SD$ , and  $S \eta$ .

If the SD-scale is placed between the other two, Equation (10) requires the following conditions: (1) All scales must be logarithmic. (2) If the lengths of the cycles on the scales  $V$ , SD and  $S\sqrt{V}$  are designated by  $a$ ,  $b$ , and  $c$ , respectively, and if the normal distances between the corresponding scales are  $AB$ ,  $AC$ , and  $BC$ , these proportions must obtain:

$$\frac{AB}{AC} = \frac{b}{2c} ; \frac{BC}{AC} = \frac{b}{2a} ; \frac{BC}{AB} = \frac{c}{a}$$

These conditions permit solution for all values within the Stokes range, without further orientation of the three axes. In order to take into consideration the fact that these scales as such are not valid beyond  $R = 0.1$ , the relative vertical positions of the several scales have been so chosen that when the straight-edge is horizontal, readings on the three scales will invariably satisfy the condition

$$R = \frac{VD}{V} = \frac{V(SD)}{(S\sqrt{V})} = 0.1$$

This then establishes the relationships

$$AC = \frac{4}{3} AB = 4 BC ; a = 2b = 3c$$

For any other value of  $R$ , the straight-edge will then slope in one direction or the other. When  $R$  is less than 0.1, for instance, the left end will always be higher, the slope increasing as  $R$  becomes smaller. Advantage was taken of this fact in the following manner: For each cycle of  $R$  a separate velocity scale was added, all points of the same velocity being connected by smooth curves. These scales were displaced to the right by such an amount that the velocity may always be read a fixed distance  $AC$  along the straight-edge from the base line - the  $S\sqrt{V}$ -scale. In this way the chart was greatly foreshortened, and means were provided of determining the magnitude of  $R$  in the one operation.

Since for all values of the Reynolds number beyond the Stokes limit the straight-edge will slope in the opposite direction, a similar method was used for Reynolds numbers greater than 0.1. It will be evident from Plate II that the form of the velocity curves is now quite different, owing to the fact that the  $\sqrt{V}$ -versus- $R$  function changes considerably as  $R$  increases. These curves correspond to experimental values plotted in Plate I; the final abrupt drop has not been included, however, since problems of free fall seldom involve such high magnitudes of  $R$ .

It will be noted that the SD-scale and the  $S\sqrt{V}$ -scale read directly in terms of  $D$  and  $\sqrt{V}$  when  $S$  is unity; for specific-gravity ratios that yield other values of  $S$ , the process of multiplication of  $D$  and  $\sqrt{V}$  by the same factor is graphically equivalent to shifting the two



scales upward or downward by proportional amounts, the straight-edge thereby rotating about a fixed point on the vertical line shown at the right of the chart. In the case of fixed scales this may be accomplished only by actual multiplication. With little trouble, however, the D-scale and the  $\nu$  -scale may be made adjustable, exactly as in the case of a slide rule. Multiplication by  $S$  is then effected by shifting the two slides according to adjacent logarithmic scales of  $S$ , the sliding scales then always reading direct. These are not indicated in Plate II, in order to avoid confusion, but they may easily be added by those interested in the method.

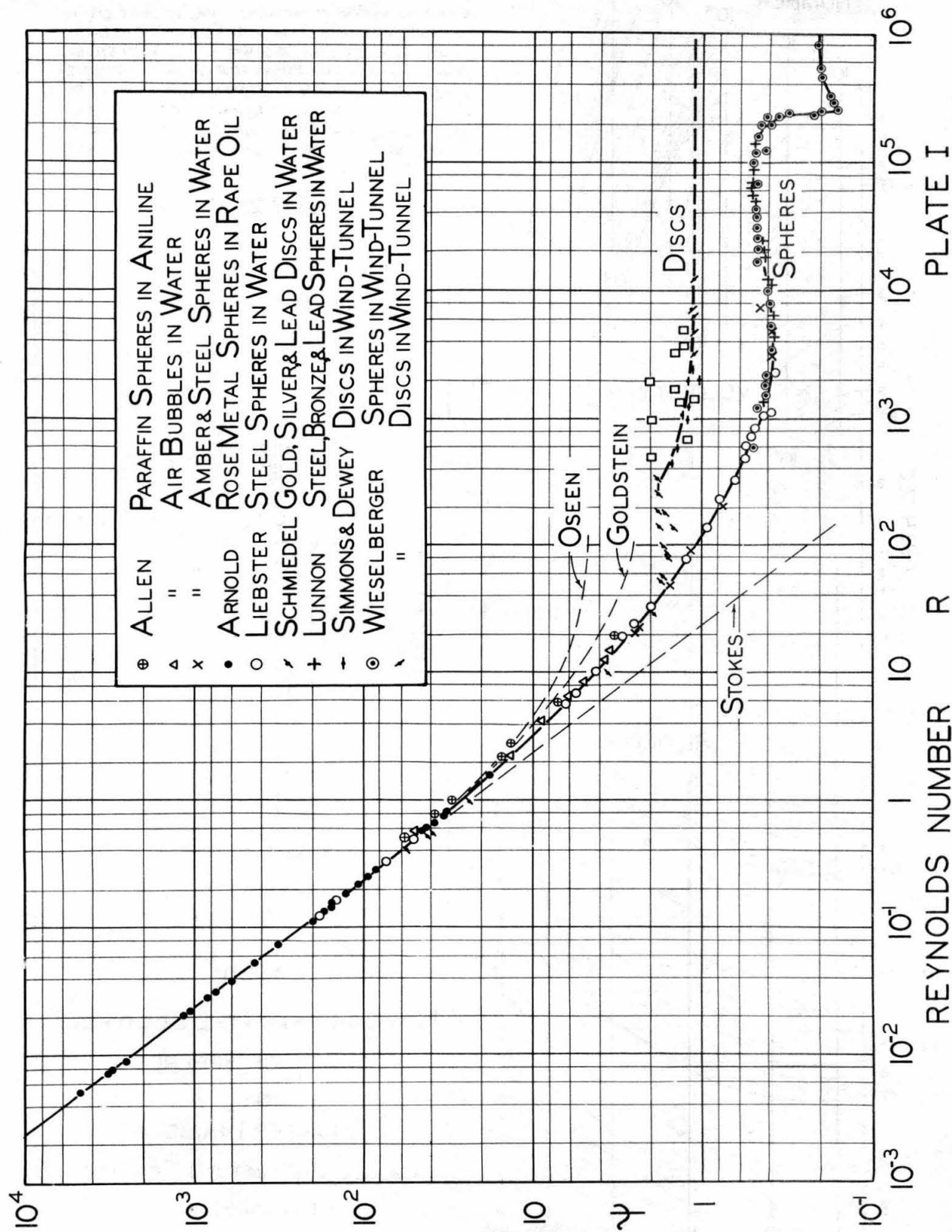
Use of the nomogram in its present form can best be illustrated by two simple numerical examples; as a general rule, both  $S$  and  $\nu$  are known values for a given problem, the determination of either  $V$  or  $D$  being required:

(1) Assume that the diameter of a sphere is 5 microns ( $5 \times 10^{-4}$  cm) and that it is desired to find its velocity of fall through water at a kinematic viscosity of  $0.01 \text{ cm}^2/\text{sec}$ . Were the specific gravity of the sphere twice that of water,  $S$  would equal unity, under which conditions the diameter and kinematic-viscosity scales would be direct reading. As indicated on Plate II, the corresponding velocity of fall, read at point A on the straight-edge, would be  $1.35 \times 10^{-3} = 0.00135 \text{ cm/sec}$ . On the other hand, for the value  $S = 10$ , the straight-edge would be shifted by the process of multiplication, the velocity then being  $0.0135 \text{ cm/sec}$ . It will be noted that since  $R$  is less than 0.1, the identical value of  $V$  may be obtained from the V-scale at the left, by projecting the line of the straight-edge.

(2) Assume that a sphere of unknown diameter is found to fall through a fluid with a terminal velocity of  $10 \text{ cm/sec}$ ; the specific gravity of the sphere is 5 times that of the fluid, and the kinematic viscosity of the fluid is found to be  $\nu = 0.15 \text{ cm}^2/\text{sec}$ . Therefore,  $S = 4$ , and  $S\nu = 0.6$ . By placing end C of the straight-edge at this point on the S-scale, and swinging end A until it passes through the line  $V = 10 \text{ cm}$ , the product of  $SD$  will be found to have the value 0.42.  $D$ , then, is slightly greater than 0.1 cm. It will be seen, moreover, that  $R$  is approximately 6, considerably above the Stokes range. The percent error involved in assuming Stokes' law to apply may be found by extending the line of the straight-edge until it intersects the velocity scale at the point  $V = 16 \text{ cm/sec}$ . The error would then be  $(16-10)100/10 = 60\%$ .

It is believed by the writer that use of this nomogram will provide a ready means of solving problems of this nature, in particular for exploratory work prior to selection of a fluid medium for experimental purposes. The investigator will undoubtedly discover a simplifying trick





or so of his own - even to a means of performing the multiplication graphically. The writer has endeavored to find such a solution, but aside from fixing the D-scale and the  $\nu$  -scale on separate slides, as already mentioned, each method proved more bothersome than the simple numerical calculations. While a nomogram of larger size would obviously permit greater accuracy, the chart is offered at the present time in this elementary form to determine the extent to which it may prove of value to the profession.

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